(4) Compound pendulum:

A rigid body capable of oscillating in a vertical plane above a fixed horizontal axis is called a compound pendulum. Let the vertical plane of oscillation be XY, O the point through which axis of rotation process, C is the centre of mass. Let mass of pendulum be m, moment of inertia about axis of rotation I and distance OC = l.

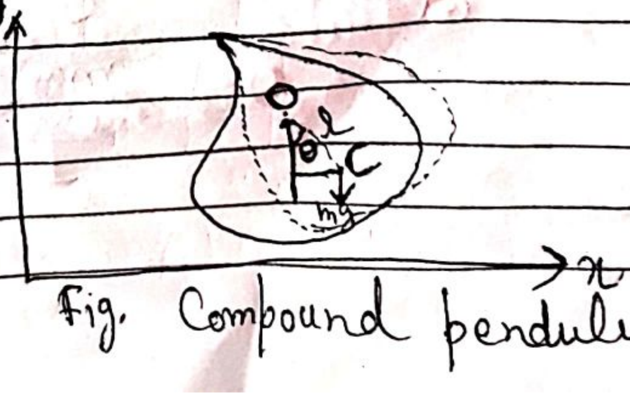

Fig. Compound pendulum

If θ be the angle through which the body is deflected, then kinetic energy is

$$T = \frac{1}{2} I \dot{\theta}^2$$

The potential energy relative to a horizontal plane through O is

$$V = -mgl \cos\theta$$

The Lagrangian

$$L = T - V = \frac{1}{2} I \dot{\theta}^2 + mgl \cos\theta$$

So that $$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$$

and, $$\frac{\partial L}{\partial \theta} = -mgl \sin\theta$$

As θ is chosen as generalised co-ordinate, Lagrangian equation of motion takes the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}(I\dot{\theta}) + mgl \cdot \sin\theta = 0$$

$$\ddot{\theta} + \frac{mgl}{I} \sin\theta = 0$$

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If amplitude of oscillation is small, then

$$\ddot{\theta} + \frac{mgl}{I}\theta = 0$$

Which is an equation for simple harmonic motion of time period

$$T = 2\pi\left(\frac{I}{mgl}\right)^{1/2}$$

Cyclic or Ignorable Co-ordinate

We have shown that Lagrangian L is a function of generalised co-ordinate q_j, generalised velocity $\dot{q}_j$ and time t. Now if the Lagrangian of a system does not contain a particular co-ordinate q_k, then obviously for such a system $\frac{\partial L}{\partial q_k} = 0$.

Such a co-ordinate is referred to as an ignorable or cyclic co-ordinate.

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Generalised momentum :

It is also termed as Conjugate or Canonical momentum.

Let us consider a system of mass points acted upon by forces derived from potentials dependent on position only. For such a system, called Conservative one, Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

Suppose q_j Co-ordinate is cyclic i.e. it does not occur in Lagrangian L, then for this Co-ordinate Lagrange's equation reduces to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

which on integration yields

$$\frac{\partial L}{\partial \dot{q}_j} = \text{Constant}$$

It is quite easy to recognise $\frac{\partial L}{\partial \dot{q}_j}$ as generalised momentum if we look at the following expression:

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i}$$

$$= \frac{\partial T}{\partial \dot{x}_i}$$

$$= \frac{\partial}{\partial \dot{x}_i} \Sigma \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

$$\frac{\partial L}{\partial \dot{x}_i} = m_i \cdot \dot{x}_i = p_{ix},$$

which is x- component of the linear momentum associated with the i^{th} particle. Generalising the concept a momentum associated with the co-ordinate q_j shall be

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j}$$

and is termed as generalised momentum. Thus

$$p_j = \text{Constant}$$

Which means that the generalised

momentum conjugate to a cyclic co-ordinate is conserved.

If we put p_j in Lagrange's equation, we get

$$\dot{p}_j = \frac{\partial L}{\partial q_j}$$